IMRN International Mathematics Research Notices
Volume 2006, Article ID 70272, Pages 1–21

Realizing 4-Manifolds as Achiral Lefschetz Fibrations

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1 Introduction

Symplectic 4-dimensional manifolds are known to be characterized as those admitting the structure of a Lefschetz fibration. More precisely, Donaldson [5] proved that every symplectic 4-manifold admits a Lefschetz pencil, which can be blown up at its base points to yield a Lefschetz fibration. Conversely, Gompf [19] showed that any 4-manifold with a Lefschetz fibration admits a symplectic structure, provided the fibers are nontrivial in homology.

The definition of a Lefschetz fibration includes the provision that the orientations in the local holomorphic description of a critical point match the global orientations of the total and base spaces of the fibration. This condition is crucial for the above results, for symplectic structures serve to orient the manifolds involved, and Donaldson and Gompf each elucidate how symplectic structures on fibers are compatible with global symplectic structures. Therefore if one relaxes this requirement, the resulting wider class of fibrations, known as *achiral* Lefschetz fibrations, will no longer respect symplectic structures. It is natural to ask which arbitrary (i.e., not necessarily symplectic) smooth manifolds admit achiral Lefschetz fibrations.

The first result concerning the existence of achiral Lefschetz fibrations is due to Harer [20] who proved that a 4-manifold that has a handle decomposition with only 0-handles, 1-handles, and 2-handles admits an achiral Lefschetz fibration over the disk. It was observed in [19] that any closed simply connected 4-manifold admits an achiral

Received 22 January 2006; Revised 17 March 2006; Accepted 23 March 2006
Communicated by S. K. Donaldson

Lefschetz fibration over S^2 after connect summing with $S^2 \times S^2$ some number of times—this number is unknown and depends on the manifold. There is no similar statement for nonsimply connected 4-manifolds.

In the other direction, the only known obstruction to the existence of an achiral Lefschetz fibration, also found in [19], is that for a manifold with positive definite intersection form, the inequality

$$1 - b_1 + b_2 \geq q \geq 0, \tag{1.1}$$

must hold, where q is the number of negative vanishing cycles. (There is an analogous result for negative definite manifolds.) Thus, for example, $\#_n S^1 \times S^3$ does not admit an achiral Lefschetz fibration for $n > 1$.

Our main result is the following.

Theorem 1.1. Let X be a smooth, closed, oriented 4-manifold. Then there exists a framed circle in X such that the manifold obtained by surgery along that circle admits an achiral Lefschetz fibration with base S^2. Moreover, all these fibrations admit sections. □

Since surgery on a circle in simply connected 4-manifolds always changes the manifold by a connected sum with an S^2-bundle over S^2, we immediately see that $X \# S^2 \times S^2$ or $X \# S^2 \widetilde{\times} S^2$ admits an achiral Lefschetz fibration whenever X is simply connected. This can be strengthened to the following result.

Corollary 1.2. Let X be a smooth, closed, simply connected 4-manifold. Then both $X \# S^2 \times S^2$ and $X \# S^2 \widetilde{\times} S^2$ admit an achiral Lefschetz fibration. □

Recently work of Taubes [26] has created a great deal of interest in near-symplectic structures. These are closed 2-forms on a 4-manifold that are symplectic off an embedded 1-manifold, and vanish in a prescribed way along this 1-manifold (see Section 7). We prove the following about achiral Lefschetz structures and near-symplectic structures.

Theorem 1.3. If a 4-manifold admits an achiral Lefschetz fibration over S^2 with a section, then it has a near-symplectic structure. Moreover, the near-symplectic structure can be chosen so that any preassigned disjoint sections are symplectic. □

We note that the achiral Lefschetz fibrations in Theorem 1.1 all admit sections and thus their intersection pairing has a hyperbolic pair. Thus $b_2^+ > 0$ for these manifolds, so the existence of a near-symplectic structure follows from a result of Honda [23]. Our proof, however, gives a more explicit construction of the near-symplectic form (cf. [15]) and illuminates the relationship between the achiral Lefschetz structure and the near-symplectic structure. This allows, among other things, for the possibility of a Donaldson-Smith approach to studying symplectic submanifolds/holomorphic curves in these manifolds, as in [6]. Combining this theorem with **Theorem 1.1** yields the following result.

Corollary 1.4. Let X be a smooth, closed, oriented 4-manifold. Then there exists a framed circle in X such that the manifold obtained by surgery along that circle admits a near-symplectic structure. Moreover, if X is simply connected, then both $X \# S^2 \times S^2$ and $X \# S^2 \widetilde{\times} S^2$ admit near-symplectic structures. □

The method of proof for **Theorem 1.3** yields a different proof of the well-known result of Gompf mentioned above.

Theorem 1.5 (Gompf and Stipsicz [19])**.** If a 4-manifold X admits a Lefschetz fibration over S^2 with a section, then it has a symplectic structure. Moreover, the symplectic structure may be chosen so that any preassigned disjoint sections are symplectic. □

This is actually weaker than Gompf's result, where one does not need to assume the existence of a section, only that the fiber is nontrivial in homology; however, we are still able to recover an important corollary of Gompf's result.

Corollary 1.6. If a 4-manifold X admits a Lefschetz pencil, then it admits a symplectic structure. □

This corollary was previously observed to follow from arguments similar to ours by Gay [14].

2 Lefschetz fibrations, open book decompositions, and handlebodies

A *Lefschetz fibration* of an oriented 4-manifold X is a map $f : X \to F$ to a surface F such that all the critical points of f lie in the interior of X and for each critical point there is an orientation preserving coordinate chart on which $f : \mathbb{C}^2 \to \mathbb{C}$ takes the form $f(z_1, z_2) = z_1 z_2$. We assume all the critical points occur on distinct fibers.

If x is a noncritical value in F, then $\Sigma = f^{-1}(x)$ is a surface properly embedded in X. The diffeomorphism type of $f^{-1}(x)$ is independent of the noncritical value x, and may have boundary, if X does. Let p be a critical point in X and U a closed disk neighborhood of $f(p)$ in F that contains no other critical values. If $y \in \partial U$ and c is a radial path in U from y to $f(p)$, then there is an embedded disk D_c in X that projects to γ and $f^{-1}(x) \cap D_c$ is a simple closed curve γ_p in the fiber above x for all $x \in (c \setminus \{f(p)\})$. (Note we use γ_p for all curves in the fibers above c.) Note that γ_p will usually be nontrivial in the homology of the fiber but will be trivial in the homology of $f^{-1}(U)$. The curve γ_p is called the *vanishing cycle* associated to p. It can be shown that $f^{-1}(U)$ is obtained from $\Sigma \times D^2$ by attaching a 2-handle to γ_p with framing one less than the framing induced on γ_p by $f^{-1}(y)$. In addition $f^{-1}(\partial U)$ is a Σ-bundle over $S^1 = \partial U$ with monodromy given by a positive Dehn twist along γ_p, which we denote by D_{γ_p}.

More generally, if $F = D^2$, then we fix a point $y \in \partial D^2$ and a collection of embedded arcs $c_1, \ldots, c_k$ connecting y to the critical points $p_1, \ldots, p_k$, such that they only intersect at y. We order the c_i's so that a small circle about y intersects them in a counterclockwise order. We now have a collection of vanishing cycles $\gamma_{p_1}, \ldots, \gamma_{p_k}$ in $\Sigma = f^{-1}(y)$. The manifold X is obtained from $\Sigma \times D^2$ by attaching 2-handles along the γ_{p_i}'s with framing one less than the framing induced by Σ. Moreover, $f^{-1}(\partial D^2)$ is a Σ-bundle over S^1 with monodromy $D_{p_1} \circ \cdots \circ D_{p_k}$.

If the fibers of the Lefschetz fibration do not have boundary (and $F = D^2$), then $\partial X = f^{-1}(\partial D^2)$ is the surface bundle described above. If the fibers do have boundary, then $\partial X = ((\partial\Sigma) \times D^2) \coprod (f^{-1}(\partial D^2))$. Clearly $(\partial\Sigma) \times D^2$ is a neighborhood of $B = \partial f^{-1}(x) \subset \partial X$ for any x in the interior of D^2. Thus it is easy to see that $(\partial X) \setminus B$ is the Σ-bundle over S^1 described above. More specifically, the Lefschetz fibration induces an *open book decomposition* of ∂X. Recall an *open book decomposition* of a 3-manifold M is a pair (B, π) where B is an oriented link in M and $\pi : M \setminus B \to S^1$ is a fibration of the complement of B such that $\partial\overline{\pi^{-1}(\theta)} = B$ for all $\theta \in S^1$. The fibers of π are called *pages* of the open book and B is called the *binding* of the open book. For a more complete discussion of topological Lefschetz fibrations and open book decompositions see [**11**, **19**].

An *achiral Lefschetz fibration* is an oriented 4-manifold X and a map $f : X \to F$ exactly as in the definition of Lefschetz fibration above except that the coordinate charts do not have to be orientation preserving. Critical points with nonorientation preserving charts will be called *negative* critical points. The entire discussion above carries over to the achiral case, except that the 2-handles attached to the vanishing cycle of a negative critical point will have framing one more than the framing induced by Σ and the contribution to the monodromy will be a left-handed Dehn twist $D^{-1}_{\gamma_p}$ about the vanishing cycle.

A key theorem for the proof of our main theorem is the following slight extension of a result of Harer.

Theorem 2.1 (Harer [20])**.** Let X be a 4-dimensional handlebody with all handles of index less than or equal to two. Then X admits an achiral Lefschetz fibration over D^2 with bounded fibers. Moreover, the vanishing cycles can be assumed to be nonnull homologous on the fiber. □

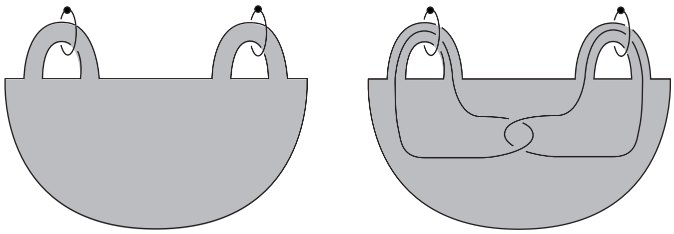

Figure 2.1 The surface Σ in X_1.

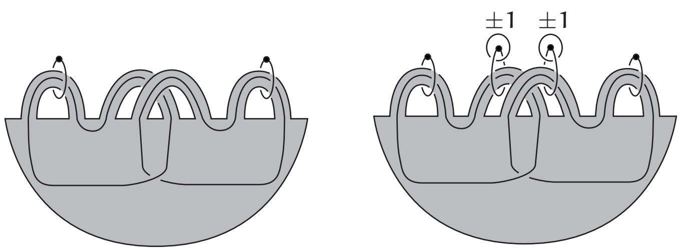

Figure 2.2

Sketch of proof. We briefly sketch Harer's original proof, since it is very nice and not easy to find in the literature (cf. [1]). Recall if Σ is any oriented surface with boundary, then $\Sigma \times D^2$ is diffeomorphic to B^4 with k 1-handles attached, where $k = -\chi(\Sigma) + 1$. Thus if X has k 1-handles, then we begin by letting Σ be a disk with k open disks removed. We can picture Σ as a disk-with-bands inside of the 1-handlebody X_1 of X as on the left side of Figure 2.1. Note that we have constructed a trivial Lefschetz structure for X_1.

We next consider the attaching link L for the 2-handles of X. We first isotope L into a neighborhood of Σ so that it projects onto Σ with only double points, as on the right side of Figure 2.1. To prove the theorem we must modify Σ so that the attaching circle of each 2-handle can be embedded on a separate fiber of $\partial(\Sigma \times D^2)$, and arrange that each is attached with framing ± 1 relative to the product framing on $\Sigma \times D^2$. Ignoring the framings momentarily, we can accomplish the former by forming the connected sum of Σ with a torus at each double point of the projection of L onto Σ; see the left side of Figure 2.2. To arrange that the new Σ is a fiber in a non-trivial Lefschetz fibration on X_1, we add canceling pairs of 1- and 2-handles (using ± 1-framed 2-handles) for each newly introduced band, as on the right of Figure 2.2. Note that the canceling 2-handle may be isotoped to lie on Σ.

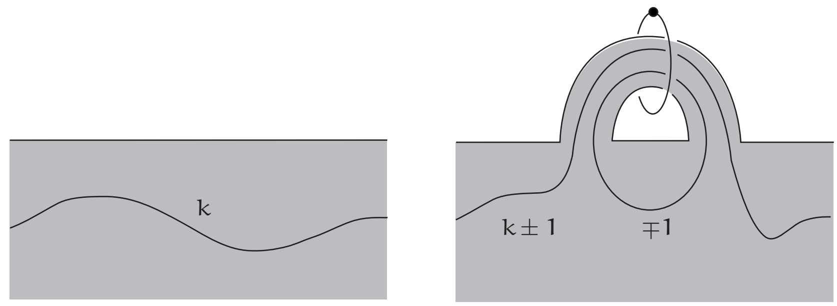

Figure 2.3 A stabilization operation for Σ.

We are left to correct the framings on the components of L. The operation shown in Figure 2.3 preserves X, and alters the framing of a component K of L by ± 1 with respect to the product framings determined by the old and new surfaces Σ. By repeating this procedure as necessary, we can arrange that the framing of K differs from the product framing by either plus or minus one. Moreover, the newly introduced ∓ 1-framed 2-handles satisfy the framing requirement as well.

We remark that by carefully analyzing this proof one can construct an achiral Lefschetz fibration for X with genus bounded above by the bridge number of the link L (as measured with respect to its projection onto Σ).

If a curve is null homologous on Σ then we may stabilize the surface and curve as in Figure 2.3. This will make the curve nonnull homologous. A further stabilization can be used to correct the framing again. ■

Let $f : X \to D^2$ be an achiral Lefschetz fibration with bounded fibers. As above we can describe X as $\Sigma \times D^2$ with 2-handles attached to vanishing cycles $\gamma_1, \ldots, \gamma_k$ with framing one less than the product framing and attached to vanishing cycles $\gamma'_1, \ldots, \gamma'_{k'}$ with framing one more than the product framing. Let Σ' be the surface obtained from Σ by attaching a 1-handle. Let γ be a simple closed curve embedded in Σ' that intersects the cocore of the new 1-handle exactly once. A *positive (negative) stabilization* of this achiral Lefschetz fibration f is the achiral Lefschetz fibration described as $\Sigma' \times D^2$ with 2-handles attached to $\gamma_1, \ldots, \gamma_k$ and $\gamma'_1, \ldots, \gamma'_{k'}$ as above and a 2-handle attached to γ with framing one less (one more) than the product framing. Note that stabilizing results in an achiral Lefschetz fibration of the same 4-manifold X. The achiral Lefschetz fibration f induces an open book decomposition (B, π) of ∂X and the positively (negatively) stabilized achiral Lefschetz fibration also induces an open book decomposition (B', π') of ∂X. The open book (B', π') is said to be obtained by positive (negative) stabilization.

3 Contact geometry and open book decompositions

An *oriented contact structure* on an oriented 3-manifold M is a hyperplane field ξ that can be written as the kernel of a 1-form α such that $d\alpha$ is nondegenerate when restricted to ξ. In other words $\xi = \ker\alpha$ and $\alpha \wedge d\alpha \neq 0$. We assume the reader is familiar with the basic notions from contact geometry (such as Legendrian knot, Thurston-Bennequin invariant, etc.). For a review, see [9]. A contact structure ξ on M is said to be supported by an open book (B, π) if ξ is isotopic to a contact structure given by a 1-form α satisfying $\alpha > 0$ on positively oriented tangents to the binding B and $d\alpha$ is a positive volume form on each page of the open book. Thurston and Winkelnkemper [27] showed that any open book supports a contact structure. In addition it is fairly simple to show that two contact structures supported by the same open book are isotopic. Recently Giroux [17] has strengthened this connection between contact structures and open book decompositions.

Theorem 3.1 (Giroux [17])**.** Let M be a closed oriented 3-manifold. There is a one-to-one correspondence between oriented contact structures on M up to isotopy and open book decompositions of M up to positive stabilization (and isotopy). □

Given a Legendrian knot L let $N(L)$ be a *standard tubular neighborhood* of the Legendrian curve L. This means the neighborhood has convex boundary and two parallel dividing curves (see [12]). Choose a framing for L so that the meridian has slope 0 and the dividing curves have slope ∞. With respect to this choice of framing, a ± 1 *contact surgery* is a ± 1 Dehn surgery, where a copy of $N(L)$ is glued to $M \setminus N(L)$ so that the new meridian has slope ± 1. Even though the boundary characteristic foliations may not exactly match up a priori, we may use Giroux's flexibility theorem [16, 21] and the fact that they have the same dividing set to make the characteristic foliations agree. This gives us a new contact manifold (M', ξ'). For a detailed discussion of contact surgery see [3]. The following is a well-known theorem; see, for example, [13].

Theorem 3.2. Suppose that L is a Legendrian knot in the contact manifold (M, ξ), ξ is supported by the open book (B, ϕ), and L is contained in a page of the open book. The contact manifold obtained from (M, ξ) by ± 1 contact surgery on L is equivalent to the one compatible with the open book with monodromy $\phi \circ D_\alpha^{\mp}$. □

Returning to Lefschetz fibrations, let $f : X \to D^2$ be an achiral Lefschetz fibration with bounded fibers. As at the end of Section 2 we can describe X as $\Sigma \times D^2$ with 2-handles attached to vanishing cycles $\gamma_1, \ldots, \gamma_k$ and $\gamma'_1, \ldots, \gamma'_{k'}$ with the appropriate framings. The Lefschetz structure on $\Sigma \times D^2$ induces an open book and hence a contact

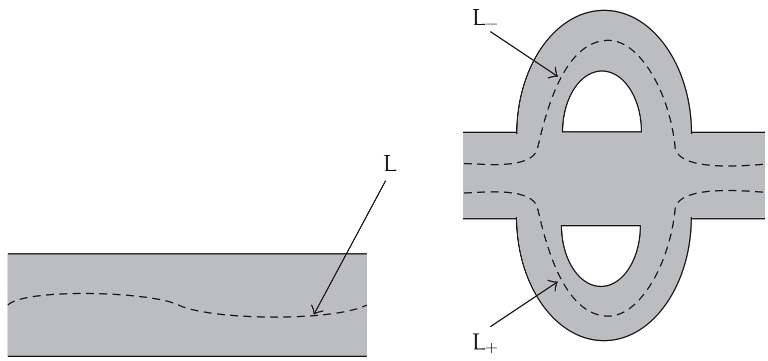

Figure 3.1 A neighborhood of a piece of L in Σ, left. (L is oriented so it points towards the left.) The twice stabilized open book, right. If the two stabilizations are positive, then $L_\pm = S_\pm(L)$ and if the stabilizations are negative, then $S_\pm(L_\pm) = L$.

structure on $\partial(\Sigma \times D^2)$. Using the Legendrian realization principle [21] we can assume the (nonnull homologous) vanishing cycles are sitting on the various pages of the open book as Legendrian curves. (Recall by Theorem 2.1 we can assume that all of the vanishing cycles are nonnull homologous.) Moreover the contact structure induced on ∂X is the one obtained from the contact structure on $\partial(\Sigma \times D^2)$ by ± 1-contact surgeries on the vanishing cycles.

In our discussion below it will be useful to see how to stabilize (and destabilize) a Legendrian knot on a page of an open book so that the stabilized knot is also on a page of an open book. Given an oriented Legendrian knot L, let $S_+(L)$ and $S_-(L)$ be the positive and negative stabilizations of L obtained by adding down or up "zigzags."

Lemma 3.3. Let (B, ϕ) be an open book decomposition supporting the contact structure ξ on M. Suppose L is a Legendrian knot in M that lies on a page of the open book. If we positively stabilize (B, ϕ) twice as shown in Figure 3.1, then we may isotop the page of the open book so that $S_+(L)$ and $S_-(L)$ appear on the page as seen in Figure 3.1.

If (B, π) is negatively stabilized twice as shown in the figure, then the contact structure supported by this open book is no longer ξ but we still see L as a Legendrian knot in the new contact structure. Moreover L_+ and L_- in the figure are now the positive and, respectively, negative destabilizations of L. That is $S_\pm(L_\pm) = L$. □

This lemma is relatively easy to prove; see [11].

4 Overtwisted contact structures and homotopy classes of plain fields

Contact structures in dimension three fall into two disjoint classes: tight and overtwisted. A contact manifold (M, ξ) is called *overtwisted* if there is an embedded disk D such that $T_x D = \xi_x$ for all $x \in \partial D$. If ξ is not overtwisted, it is called tight. One may easily prove [11] that if one negatively stabilizes an open book, then the associated contact structure is overtwisted. We have the following fundamental theorem of Eliashberg.

Theorem 4.1 (Eliashberg [7]). If two overtwisted contact structures are homotopic as plane fields, then they are isotopic as contact structures. □

Using this theorem we can understand overtwisted contact structures by understanding their homotopy classes of plane fields. According to [18] the homotopy class of an oriented plane field ξ on M is completely determined by two invariants. To simplify the discussion we will assume $H^2(M;\mathbb{Z})$ has no 2-torsion (this will suffice for our applications). The first invariant is the first Chern class (a.k.a. Euler class) $c_1(\xi) \in H^2(M;\mathbb{Z})$, which is simply the obstruction to the existence of a nonzero section of ξ. Suppose the contact manifold (M, ξ) is supported by an open book (B, π) that is induced as the boundary of the achiral Lefschetz fibration $f : X \to D^2$. We describe this Lefschetz fibration as in Section 2 with vanishing cycles $\gamma_1, \dots, \gamma_k$ and $\gamma'_1, \dots, \gamma'_{k'}$. We can assume the γ_i and γ'_i are Legendrian knots in $\partial\Sigma \times D^2$. A slight generalization of a formula from [18] (see [25]) computes the Poincaré dual to $c_1(\xi)$ as

$$\text{P.D.}\, c_1(\xi) = \sum_{i=1}^{k} r(\gamma_i) c_i + \sum_{i=1}^{k'} r(\gamma'_i) c'_i, \tag{4.1}$$

where the c_i and c'_i are the images of the cocores of the 2-handles attached to the γ_i and γ'_i's under the boundary map in the long exact sequence of the pair (X, M).

The second invariant of a homotopy class of oriented plane fields is the so-called 3-dimensional invariant $d_3(\xi)$, which is a rational number well defined modulo the divisibility of $c_1(\xi)$. We will only describe how to compute $d_3(\xi)$ when $c_1(\xi) = 0$. To this end let M and X be as above. Then we have

$$d_3(\xi) = \frac{1}{4}\big(c^2(X) - 3\sigma(X) - 2\chi(X)\big) + q, \tag{4.2}$$

where σ is the signature of X, χ is the Euler characteristic, and q is the number of negative vanishing cycles of X. The number $c^2(X)$ is the square of the class $c(X)$ with Poincaré dual

$$\sum_{i=1}^{k} r(\gamma_i) C_i + \sum_{i=1}^{k'} r(\gamma'_i) C'_i, \tag{4.3}$$

where the C_i and C_i''s are the cocores of the 2-handles attached along γ_i and γ_i'. Note that $c(X)|_M = c_1(\xi)$, which we are assuming to be zero. Thus $c(X)$, which naturally lives in $H^2(X;\mathbb{Z})$, comes from a class in $H^2(X,\partial X;\mathbb{Z})$ and thus can be squared. Formula (4.2) is a slight generalization of the one given in [3], where it was assumed that X had no 1-handles. Their proof carries over to our case. In particular, according to [18], $d_3(\xi) = (1/4)(c^2(Y) - 3\sigma(Y) - 2\chi(Y))$ where Y is any almost complex 4-manifold with $M = \partial Y$ and ξ is the set of almost complex tangencies to M. If X is as above, then there is a natural almost complex structure on $Y = X \#_q \overline{\mathbb{C}P}^2$ (see [3]) with ξ the set of complex tangencies. Moreover,

$$c_1(Y) = c(X) + (3, \ldots, 3) \in H^2(X;\mathbb{Z}) \oplus_q H^2\left(\overline{\mathbb{C}P}^2;\mathbb{Z}\right), \tag{4.4}$$

$\sigma(Y) = \sigma(X) + q$, and $\chi(Y) = \chi(X) + q$. The formula follows.

5 Construction of global Lefschetz fibrations

We are now ready to establish the existence of Lefschetz fibrations.

Proof of Theorem 1.1. We begin by giving X an arbitrary handlebody structure, letting X_1 denote the union of the 0-, 1- and 2-handles of X, and X_2 denote the union of the 3- and 4-handles. Then as each X_i is a 2-handlebody, we can use Theorem 2.1 to find achiral Lefschetz fibrations (with bounded fibers) $f_1 : X_1 \to D^2$ and $f_2 : X_2 \to D^2$, with each inducing an open book structure on the common boundary $\partial X_1 = -\partial X_2$. We can stabilize the achiral Lefschetz fibrations so that each has fibers with connected boundary.

If these induced open books are the same (under the identification of ∂X_1 and $-\partial X_2$ used to reconstruct X), we can attempt to reconstruct X from the pieces X_1 and X_2 by gluing them along their boundaries in a two-step process. We first glue along the pages of the open books by forming

$$W = X_1 \bigcup_{f_1^{-1}(\partial D^2) = f_2^{-1}(\partial D^2)} X_2. \tag{5.1}$$

We then have an achiral Lefschetz fibration with bounded fibers

$$f_1 \cup f_2 : W \longrightarrow S^2. \tag{5.2}$$

Since the fibers in the two achiral Lefschetz fibrations have connected boundary we see $\partial W = S^1 \times S^2 = S^1 \times D_1^2 \cup S^1 \times D_2^2$, where $S^1 \times D_i^2$ is $\partial X_i \setminus f_i^{-1}(\partial D^2)$. Gluing $S^1 \times D_1^2$ to $S^1 \times D_2^2$ in ∂W will yield X. Gluing an $S^1 \times D^3$ to W will produce the same result as gluing

$S^1 \times D^2_1$ to $S^1 \times D^2_2$. So we see that

$$X = W \cup S^1 \times D^3 \tag{5.3}$$

or said another way there is an embedded curve γ in W such that $W = X \setminus N$ where N is an open tubular neighborhood of γ.

Notice that in a collar of $\partial W = S^1 \times S^2$ we may express the above achiral Lefschetz fibration as the projection $I \times S^1 \times S^2 \to S^2$. If we now glue in $D^2 \times S^2$ so that each $\partial D^2 \times \{\text{pt.}\}$ matches to $S^1 \times \{\text{pt.}\}$, then the resulting closed manifold has an achiral Lefschetz fibration over S^2. Moreover, this manifold is the result of surgering X along the circle γ.

The theorem is therefore proven once we establish the following proposition. ■

Proposition 5.1. Let X be a closed, smooth, 4-manifold. Then X may be decomposed as $Y_1 \cup Y_2$, where each Y_i is a 2-handlebody which admits an achiral Lefschetz fibration over D^2 with bounded fibers of the same genus, and with the induced open books on $\partial Y_1 = -\partial Y_2$ coinciding. □

Proof. Fix a handle decomposition of X and let Y_1 be the union of the 0-, 1-, and 2-handles and let Y_2 be the union of the 3- and 4-handles. By Harer's Theorem 2.1 we know there are achiral Lefschetz fibrations $f_i : Y_i \to D^2$, $i = 1, 2$, with bounded fibers. By adding a canceling 2-handle/3-handle pair to X if necessary we may assume that $-Y_2$ has one 0-handle and an even number $2k$ of 1-handles. We may then write $-Y_2$ as $\Sigma \times D^2$ where Σ is a genus k surface with one boundary component. Written as such, Y_2 has an obvious Lefschetz fibration with no singular fibers. We take f_2 to be this fibration.

Let ξ_i be the contact structure supported by the open book associated to the achiral Lefschetz fibration f_i, $i = 1, 2$. By Giroux's Theorem 3.1 these open books associated to the achiral Lefschetz fibrations will be isotopic, after possible positive stabilization, if the supported contact structures are isotopic. We will show how to choose the achiral Lefschetz fibrations so that the associated contact structures ξ_1 and ξ_2 are isotopic. We begin by showing they are homotopic as plane fields. To this end notice that $-Y_2$ supports a Stein structure and hence ξ_2 is tight. In addition using (4.1), we see $c_1(\xi_2) = 0$.

Let Y_1' denote the union of the 0- and 1-handles of Y_1. Let $K_1, \dots, K_l \subset \partial Y_1'$ be the attaching circles for the 2-handles in Y_1. We know there is a Lefschetz fibration of Y_1' so that the K_i are Legendrian in the contact structure supported by the induced open book. Moreover we can assume the K_i lie on distinct pages of the open book and the attaching framing differs from the contact framing by ± 1 the page framing (= contact framing). Attaching the 2-handles now gives a natural achiral Lefschetz fibration to Y_1.

By (4.1) the Poincaré dual of the first Chern class of the induced contact structure ξ_1 is

$$\text{P.D.}\, c_1(\xi_1) = \sum_{i=1}^{l} r(K_i) c_i, \tag{5.4}$$

where $c_i \in H_1(\partial Y_1; \mathbb{Z})$ is the image of the cocore of the ith 2-handle under the boundary map in the long exact sequence of the pair $(Y_1, \partial Y_1)$.

In [18] it was shown that the parity of $r(K_i)$ is fixed by the contact framing of K_i and the number of 1-handles K_i runs over. We claim that it is possible to alter the Lefschetz fibration (and hence the contact structure) so that any integer with the right parity can be realized as the rotation number of K_i. We begin by stabilizing the Lefschetz fibration of Y_1' one time positively and one time negatively. This does not effect the Chern class of the contact structure induced on $\partial Y_1'$, though the contact structure is different. Thinking of K_i as a Legendrian knot on a page of this new open book for $\partial Y_1'$, let K_i' be the result of pushing K_i over the two new 1-handles in the page. We can assume K_i' is Legendrian. Using Lemma 3.3 we see that the framings on K_i and K_i' coming from the page are the same. Moreover, using Lemma 3.3, we can choose the stabilizing 1-handles so that $r(K_i') = r(K_i) \pm 2$. Thus by a sequence of stabilizations we can alter the rotation number of any K_i by any even number. Since $\text{P.D.}\, c_1(\xi_1)|_2 = 0$ (see [19]), it now follows that there exists a sequence of alterations of rotation numbers which gives $c_1(\xi_1) = 0$. (To see this, let G denote the subgroup of $H_1(\partial Y_1)$ generated by $c_1, \dots, c_l$, and note that the subgroup of even elements of G is generated by $2c_1, \dots, 2c_l$. Hence we may write $\text{P.D.}\, c_1(\xi_1) = \sum_{i=1}^{l} a_i(2c_i)$, which combined with (4.1) gives $\sum_{i=1}^{l}(r(K_i) - 2a_i)c_i = 0$.)

At this point we may assume that $c_1(\xi_1) = c_1(\xi_2) = 0$. For a manifold with no 2-torsion in the second cohomology, as is the case here, the homotopy class of a plane field with $c_1 = 0$ is determined by the invariant d_3. If ξ is supported by an open book and ξ' by the open book obtained by negatively stabilizing one time, then (4.2) yields $d_3(\xi') = d_3(\xi) + 1$. By negatively stabilizing the achiral Lefschetz fibration on Y_1 or Y_2 we may assume that $d_3(\xi_1) = d_3(\xi_2)$ and thus ξ_1 and ξ_2 are homotopic as plane fields. If we now negatively stabilize the achiral Lefschetz fibrations on each of Y_1 and Y_2, we may assume the associated contact structures are overtwisted. Eliashberg's Theorem 4.1 allows us to conclude that ξ_1 and ξ_2 are isotopic contact structures. ■

Proof of Corollary 1.2. Let X be a simply connected 4-manifold and γ the curve identified in the proof of Theorem 1.1 on which surgery produces a manifold X' with an achiral Lefschetz fibration over S^2. It is well known (see, e.g., [19]) that X' is either $X \# S^2 \times S^2$ or

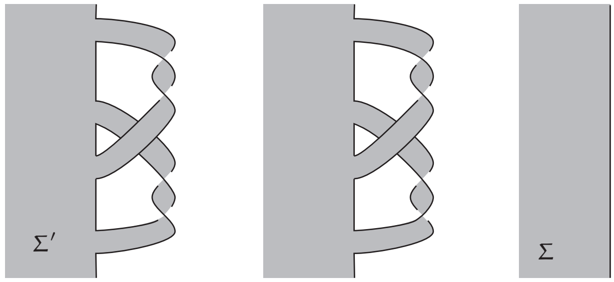

Figure 5.1 Surface Σ' on the left. In the middle we have $\partial\Sigma'$ with a crossing change and on the right there is Σ. The boundary of the middle surface is isotopic to the boundary of the right-hand surface. Although the surfaces are not isotopic, they give the same framing to the knot.

$X \# S^2 \widetilde{\times} S^2$, with the outcome determined by the framing of γ. (The set of framings can be identified with $\pi_1(SO(3)) = \mathbb{Z}_2$.) Moreover, since $\gamma \subset M^3 \subset X^4$, a framing of γ in M gives a framing of γ in X. Recall that γ is the binding of an open book and the framing of γ in M comes from a page Σ in the open book. If we positively stabilize the open book twice as shown in Figure 5.1, we have a new knot γ' and page Σ'. The knots γ' and γ are homotopic in M (just change one crossing of γ') and hence isotopic in X. The homotopy from γ' to γ takes the framing on γ' coming from Σ' to one less than the framing on γ coming from Σ. Thus the framing on γ' in X differs from the framing on γ in X and therefore surgery on one of these curves will yield $X \# S^2 \widetilde{\times} S^2$ while surgery on the other will yield $S^2 \times S^2$. ∎

6 Symplectic and near-symplectic structures

Let X be a 4-manifold. If we fix a metric g on X, we can consider the bundle Λ^2_+ of self-dual 2-forms on X. A closed 2-form ω on X is a *near-symplectic structure* if $\omega^2 \geq 0$ and there is a metric g such that ω is harmonic and transverse to the 0-section of Λ^2_+. By transversality one can see that the zeros Z of ω^2 form a union of embedded circles. Honda [22] showed that each component of Z has a neighborhood $S^1 \times B^3$ where ω can be written as one of two models. The "orientable" model is $dt \wedge dh + \star_3 dh$, where h is a Morse function on B^3 with one index 1 or index 2 critical point at 0 and $\star_3$ is the 3-dimensional Hodge star operator. The "non-orientable" model is a $\mathbb{Z}_2$ quotient of the above model. One may define a near-symplectic structure without regard to a metric by demanding that ω is closed and symplectic away from a union of circles Z, and near each component of Z has a model as above. There has been great interest in near-symplectic structures following work of Taubes [26] that suggests they might be used to give a "geometric" understanding of Seiberg-Witten theory.

If X is allowed to have boundary, it is possible that the near-symplectic form ω degenerates along properly embedded arcs in X. In this article we will assume that Z is always a union of circles in the interior of X. Given this we can discuss the convex boundary of a near-symplectic 4-manifold (X, ω). We say ∂X is *convex*, or strongly convex, if there is a vector field ν defined near ∂X, transverse to ∂X, whose flow expands ω, namely,

$$L_\nu \omega = c\omega, \tag{6.1}$$

where L denotes Lie derivative and c is a positive constant. The 1-form $\alpha = (\iota_\nu \omega)_{\partial X}$ is a contact form on ∂X. Setting $\xi = \ker \alpha$, we will say (X, ω) is a near-symplectic filling of $(\partial X, \xi)$. It is a standard fact, see [11], that if β is any other contact form for ξ then there is a neighborhood of ∂X in X that is symplectomorphic to a (one-sided) neighborhood of the graph of some function in $(\partial X) \times \mathbb{R}$ with symplectic form $d(e^t \beta)$, where t is the coordinate on $\mathbb{R}$.

The following result, restated for our context, is well known.

Theorem 6.1 (Eliashberg [7] and Weinstein [28])**.** Suppose (X,ω) is a near-symplectic filling of (M, ξ), and L is a Legendrian knot in (M, ξ). If a 2-handle is attached along L with contact framing -1, then ω may be extended over the 2-handle to obtain a near- symplectic filling of (M', ξ'), where ξ' is the contact structure obtained by contact -1 surgery on L. Moreover, (M', ξ') is strongly convex. There are no new circles of degeneration in the extended near-symplectic structure. □

We now turn to establishing a version of this theorem for $+1$-framed surgeries. To this end we first observe an alternate description of $+1$-contact surgery. This theorem is a slight reformulation of a similar result in [13]; see also [4, 10].

Theorem 6.2. Let L be a Legendrian knot in a contact manifold (M, ξ). The contact structure obtained from ξ by performing a $+1$-contact surgery on L is the same as the contact structure obtained by performing a Lutz twist on the positive transverse push-off of L followed by a Legendrian surgery. □

We recall the definition of a Lutz twist. If γ is a knot transverse to the contact planes of (M, ξ), then γ has a standard neighborhood contactomorphic to a neighborhood of the image of the z-axis in $\mathbb{R}^2/\sim$, where $(r, \theta, z) \sim (r, \theta, z+1)$, with contact structure $dz + r^2 d\theta$. We can identify slopes $s \in \mathbb{R} \cup \{\infty\}$ of a linear foliation of T^2 by angles

$\theta_s \in \mathbb{R}/\pi\mathbb{Z}$. To distinguish different amounts of twisting we will lift θ_s to $\mathbb{R}$. We can express a neighborhood of γ as $N = S^1 \times D^2$ and assume that on concentric tori $T_a = \{r = a\}$ the characteristic foliation is linear with monotonically decreasing (as a increases) slope ranging in $(0, -\epsilon]$. (This description uniquely determines the contact structure on N.) If we leave the contact structure ξ the same outside N but change it so that the slopes of the characteristic foliation on the T_a range in $(0, -\pi-\epsilon]$, then we get a well-defined contact structure ξ' on M. This contact structure is said to be obtained from ξ by a (half-) Lutz twist along γ.

Proof. Consider a Legendrian knot L in (M, ξ). Let $N(L)$ be a standard neighborhood of L. (See Section 3.) We pick a framing on $N(L)$ so that the dividing curves on $\partial N(L)$ have slope ∞ and the meridian has slope 0. Let L_+ be the positive transverse push-off of L contained in $N(L)$, and let ξ' denote the contact structure that is the result of performing a Lutz twist on L_+. The contact structure ξ' agrees with ξ outside $N(L)$ and on $N(L)$ the contact structure ξ is determined by the fact that the slopes of the characteristic foliation on concentric tori range in $(0, -\pi/2]$, and ξ' by the fact that the slopes range in $(0, -3\pi/2]$. Inside the neighborhood $N(L)$ we may find a standard neighborhood of a Legendrian knot with twisting 2 in our chosen framing as follows. Break $N(L)$ into two pieces $N_1 \cup N_2$ where N_1 is a solid torus containing L_+ with slope ranging in $(0, -5\pi/6]$ and $N_2 = T^2 \times [0, 1]$ with slopes ranging in $[-5\pi/6, -3\pi/2]$. (So $N(L)$ is split along a torus with dividing slope $1/2$.) The solid torus N_1 is a standard neighborhood of a Legendrian knot L' with twisting number 2 with respect to the framing chosen on $N(L)$ (see [12]). Thus if we perform a Legendrian surgery on L' this will, topologically, correspond to a $+1$ surgery on L. Moreover, one can check that the contact structure on $N(L)$, after Legendrian surgery on L', has slopes of the characteristic foliation on concentric tori ranging in $(-3\pi/4, -3\pi/2]$. Such a contact structure on a solid torus is tight. Thus we have removed $N(L)$ from (M, ξ) and reglued it with a $+1$-twists and extended $\xi|_{M \setminus N(L)}$ to the surgered manifold so that it is tight on the surgery torus. This is precisely a $+1$-contact surgery. ■

To perform a Lutz twist via a near-symplectic cobordism we recall the following result.

Theorem 6.3 (Gay and Kirby [15]). Let (M, ξ) be a contact manifold and let ξ' be obtained from the contact structure ξ by a Lutz twist along the transverse curve γ. Assume the Lutz twist occurred in the neighborhood N of γ. If (X, ω) is a near-symplectic filling of (M, ξ), then ω may be extended over $X \cup M \times [0, 1]$, where ∂X and $M \times \{0\}$ are identified to be a near-symplectic filling of (M, ξ'). Moreover, ω is symplectic on $(M \setminus N) \times [0, 1]$ and ω has one singular circle in $N \times [0, 1]$. □

This theorem, coupled with Theorem 6.1 and the proof of Theorem 6.2 proves the following result.

Theorem 6.4. Suppose (X, ω) is a near-symplectic filling of (M, ξ), and L is a Legendrian knot in (M, ξ). If a 2-handle is attached along L with contact framing $+1$, then ω may be extended over the 2-handle to obtain a near-symplectic filling of (M', ξ'), where ξ' is the contact structure obtained by contact $+1$ surgery on L. Moreover, (M', ξ') is strongly convex. There is exactly one singular circle of ω in the near-symplectic structure on the 2-handle. □

We now rephrase [8, Theorem 1.1] for our current purposes.

Theorem 6.5 (Eliashberg [8]). Let (X', ω) be a near-symplectic filling of (M, ξ) and (B, π) an open book decomposition supporting the contact structure ξ. Let X be X' with 2-handles attached to B with framing given by the pages of the open book. One may extend the near-symplectic structure on X' to X so that no new circles of degeneration are added and so that the near-symplectic structure restricted to each fiber in the surface bundle ∂X is symplectic. □

The proof in [8] goes through unchanged—in fact, the proof can be simplified since we are assuming ∂X is strongly convex.

This theorem points out the need to study symplectic surface bundles. A *symplectic bundle over* S^1 is a 3-manifold M that fibers over the circle together with a closed 2-form ω which is positive on each fiber. The kernel of ω defines a line field that is transverse to the fibers of the fibration. An orientation on M and on the fibers induces an orientation on the line field and thus we can fix a fiber Σ_0 of the fibration and use the line field to define a return map $H_{(M,\omega)} : \Sigma_0 \to \Sigma_0$, called the *holonomy* of the symplectic fibration. The holonomy $H_{(M,\omega)}$ is a symplectomorphism of $(\Sigma_0, \omega|_{\Sigma_0})$. If we normalize ω so that it integrates to 1 on each fiber of the fibration, then the holonomy determines (M, ω) up to fiber preserving diffeomorphism.

Recall a symplectomorphism f of a surface (Σ, ω) is *Hamiltonian* if there are functions $H_t : \Sigma \to \mathbb{R}$ such that the vector field X_t determined by

$$\iota_{X_t}\omega = dH_t \tag{6.2}$$

generates a flow whose map at $t = 1$ is f. A Hamiltonian diffeomorphism is always isotopic to the identity, although the converse is false. The main result we need is the following.

Lemma 6.6 (Eliashberg [8]). Suppose the holonomy of the symplectic fibration (M, ω) is a Hamiltonian diffeomorphism. Then there is a symplectic form Ω on $X = \Sigma \times D^2$, such that $\partial(X, \Omega) = (M, \omega)$, where Σ is the fiber of the fibration. Moreover, it can be assumed that $\Sigma \times \{pt\}$ and $\{pt\} \times D^2$ are symplectic submanifolds in X. □

Since not all symplectomorphisms isotopic to the identity are Hamiltonian, we will need a criterion below to determine when a symplectomorphism is Hamiltonian. For this, we consider the flux map. Let $\phi_t, 0 \leq t \leq 1$, be a path of symplectomorphisms of a surface (Σ, ω). There is a family of vector fields X_t determined by

$$X_t\big(\phi_t(x)\big) = \frac{d\phi_t(x)}{dt}. \tag{6.3}$$

The flux of the path ϕ_t is defined to be

$$\text{Flux}\big(\{\phi_t\}\big) = \int_0^1 \big[\iota_{X_t}\omega\big]\,dt, \tag{6.4}$$

where $[\cdot]$ denotes cohomology class. This is an element in $H^1(\Sigma; \mathbb{R})$. (If Σ is a torus, then the flux is in $H^1(\Sigma; \mathbb{R}/\mathbb{Z})$.) One can show that the flux only depends on the path of symplectomorphisms up to homotopy with fixed endpoints. A path of symplectomorphisms is homotopic to a Hamiltonian path if and only if its flux is zero, see [2]. There is an alternate interpretation of flux (see [24]) that will be more useful below. Let (M, ω) be a symplectic bundle over S^1 with a holonomy map isotopic to the identity; assume also that the fiber Σ has genus greater than 1. In this case we can identify M as $\Sigma \times S^1$ (up to fiber preserving isotopy). Under this identification we have a map from $H_1(\Sigma)$ to $H_2(M)$ that sends $[c]$ to $[c \times S^1]$. With this understood the flux of the holonomy is the map $H_1(\Sigma) \to \mathbb{R}$ given by

$$\text{Flux}\big([c]\big) = \int_{c \times S^1} \omega. \tag{6.5}$$

7 Near-symplectic structures and achiral Lefschetz fibrations

We are now ready to prove our main theorem concerning near-symplectic structures.

Proof of Theorem 1.3. Suppose $f : X \to S^2$ is an achiral Lefschetz fibration with fiber Σ and section S. Let N_Σ be a neighborhood of Σ that is fibered by nonsingular fibers of f and let N_S be a neighborhood of S that contains no singular points of f. Let $X' = X \setminus (N_\Sigma \cup N_S)$. We can describe X' as an achiral Lefschetz fibration over D^2 with nonsingular fibers Σ' by restricting f to X'. Here Σ' is simply Σ with an open disk removed. Thus X' may be built from $\Sigma' \times D^2$ by attaching 2-handles along curves $\gamma_1, \ldots, \gamma_k$ and $\gamma'_1, \ldots, \gamma'_{k'}$ on fibers with framing one less and, respectively, one more than the fiber framing. We know $\Sigma' \times D^2$ has a symplectic structure with convex boundary, see [11]. Moreover, the open book induced on the boundary from the product structure supports the induced contact structure on the boundary. Thus from Theorems 6.1 and 6.4, we see that X' has a near-symplectic structure, with one degenerate circle for each γ_i'. In addition, with this near-symplectic structure the boundary of X' has a contact structure supported by the open book induced by f.

Notice that $N_\Sigma \cup N_S$ is simply $\Sigma \times D^2$ union a 2-handle h. If we view h as attached to X' instead of N_Σ, it will be attached to the binding of the open book with framing coming from the fibers of the open book. This process simply caps off the fibers Σ' to recover the fibers Σ, since the result of moving h to X' is $X \setminus (\Sigma \times D^2)$ which has an achiral Lefschetz fibration over D^2 with fiber Σ. Thus the result of surgery along the binding of the open book for $\partial X'$ with framing coming from the fiber is $\Sigma \times S^1$. Moreover, by Theorem 6.5, we know the near-symplectic form on X' extends to $X' \cup h$ so that the surfaces $\Sigma \times \{pt\}$ are all symplectic. If the holonomy of this symplectic fibration is trivial, or Hamiltonian isotopic to the identity, then, by Lemma 6.6, we may extend this near-symplectic structure over N_Σ thus constructing the near-symplectic structure on X. We are left to show that we can arrange the holonomy to be trivial.

We begin by observing that the monodromy of the open book is a composition of Dehn twists parallel to the boundary of Σ'. (Of course the monodromy expressed in terms of Dehn twists along the γ_i's and γ_i''s might look more complicated, but it will be isotopic to this.) We may assume that the monodromy is supported in a collar neighborhood of the boundary, and we write Σ'' for the complement of this neighborhood in Σ'. The complement in $M = \partial X'$ of the binding and the support of the monodromy (in all fibers) can be written as $\Sigma'' \times S^1$, and is denoted by M'. The contact structure ξ on M' is isotopic to one given by the kernel of $\alpha|_{M'} = K dt + \lambda$, where K is any large positive constant, t is the coordinate on S^1, and λ a primitive for a volume form on Σ''. It is easy to see the Reeb vector field for α is $X = \partial/\partial t$. Now if we consider the 4-manifold $Y = M \times [a, b]$ with symplectic form $\omega = d(e^s \alpha)$, where s is the coordinate on the interval factor, the upper boundary of Y is convex and induces the contact structure ξ. In addition, the kernel of $\omega|_{M\times\{a\}}$ is spanned by the Reeb vector field and the flow of the Reeb vector field induces the identity return map on the Σ'' part of a page of the open book. Now if we attach a 2-handle to Y along the binding of the open book in $M \times \{b\}$ as in Theorem 6.5, then we obtain a symplectic manifold Y' with an upper boundary $\Sigma \times S^1$, which has symplectic fibers $\Sigma \times \{pt\}$. Since the symplectic structure is only affected near the attaching region for 2-handle, the kernel of the symplectic form restricted to the upper boundary will still induce the identity map on the Σ'' part of the fiber. Given any primitive homology class $h \in H_1(\Sigma; \mathbb{Z})$ we can represent it by an embedded curve c contained in Σ''. Now ω

restricted to $c \times S^1$ in the upper boundary of Y' is zero (since the $\{pt\} \times S^1$ is in the kernel of $\omega|_{\partial Y'}$). Thus $\int_{c \times S^1} \omega = 0$. Using our second interpretation of flux we see the flux of the holonomy is zero and hence the holonomy map is Hamiltonian isotopic to the identity.

Since $\partial X'$ is strongly convex (and the near-symplectic structure is symplectic there), a neighborhood of $\partial X'$ is symplectomorphic to a neighborhood of the graph of a function g in $M \times \mathbb{R}$ with symplectic structure $\omega = d(e^s \alpha)$. Let b be any number larger than the maximum of g. We can add the collar $\{(x, s) \in M \times \mathbb{R} : g(x) \leq s \leq b\}$ to X' and extend the symplectic structure over it so that a neighborhood of $\partial X'$ is symplectomorphic to $M \times [a, b]$, as in the previous paragraph. We may now attach the 2-handle h to X' and see the holonomy is Hamiltonian isotopic to the identity, as described above.

Finally we note that the section of X' union h given by the cocore of the 2-handle is symplectic. By Lemma 6.6, this section may be symplectically extended over N_Σ, showing that the original section of X over S^2 is symplectic. If we started with more than one section of X, then we could have removed N_Σ and neighborhoods of each of these sections to form X'. The argument above applies equally well to this case. ■

Acknowledgments

The first author was partially supported by NSF Career Grant (DMS-0239600) and FRG-0244663. The second author was partially supported by NSF Grant DMS 00-72264.

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